

INTEGRABILITY CASES FOR THE ANHARMONIC OSCILLATOR EQUATION

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Abstract

Using Euler's theorem on the integrability of the general anharmonic oscillator equation [1], we present three distinct classes of general solutions of the highly nonlinear second order ordinary differential equation

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$\frac{d^2x}{dt^2} + f_1(t) \frac{dx}{dt} + f_2(t)x + f_3(t)x^n = 0$. The first exact solution is obtained from a particular solution of the point transformed equation $d^2X/dT^2 + X^n(T) = 0$, $n \in \{-3, -1, 0, 1\}$, which is equivalent to the anharmonic oscillator equation, if the coefficients $f_i(t)$, $i = 1, 2, 3$ satisfy an integrability condition. The integrability condition can be formulated as a Riccati equation for $f_1(t)$ and $\frac{1}{f_3(t)} \frac{df_3}{dt}$, respectively. By reducing the integrability condition to a Bernoulli type equation, two exact classes of solutions of the anharmonic oscillator equation are obtained.

1. Introduction

The anharmonic oscillator is a physical system generalizing the simple linear harmonic oscillator $\frac{d^2x}{dt^2} + \omega_0^2 x(t) = 0$, where $x(t)$ is the position coordinate, t is the time, and ω_0 is the oscillation frequency. In general, the time evolution of the space variable x of the anharmonic oscillator is governed by the following nonlinear second order differential equation [1, 2]:

$$\frac{d^2x}{dt^2} + f_1(t) \frac{dx}{dt} + f_2(t)x + f_3(t)x^n = f_4(t), \quad (1)$$

where $f_i(t)$, $i = 1, 2, 3, 4$, and x are arbitrary real functions of t defined on a real interval $I \subseteq \mathbb{R}$, with $f_i(t)$ and $x(t) \in C^\infty(I)$. The factors $f_i(t)$ are physically interpreted as follows: $f_1(t)$ is a damping factor; $f_2(t)$ is a time dependent oscillation frequency coefficient; $f_3(t)$ is the simplest possible anharmonic term; $f_4(t)$ is a forcing term; and n is a real constant [2]. The equation of motion of the anharmonic oscillator is strongly nonlinear, and when the anharmonicity term $f_3(t)x^n$ is small, its solutions can be obtained by using perturbation theory. If the anharmonicity is large, then other numerical techniques need to be implemented.

The anharmonic oscillator equation (1) with specific values of the exponent n can be used to model many different physical systems. For $n = 3/2$, one obtains the Thomas and Fermi atomic model [3, 4], while the case $n = -3$ corresponds to the Ermakov [5], or Pinney [6], equation. For $n = -1$, one obtains the Brillouin electron beam focusing system equation [7, 8], and $n = 3$ gives the case of the Duffing oscillator [9].

An interesting particular case of the general anharmonic oscillator equation (1) is the Ermakov-Pinney equation (EPE), which is a well-known example of a nonlinear second order differential equation with important physical applications (we refer the reader to [10, 11] for a historical development and an excellent review of the properties of the EPE equation). The EPE is endowed with a wide range of physical applications, including quantum cosmology [12], dynamics of scalar field cosmologies and the braneworld scenario [13], quantum field theory [14, 15], nonlinear elasticity [16], nonlinear optics [17, 18], description of the wave function of Bose-Einstein condensates (BEC) at the mean-field level [19], the envelope of the electric field in nonlinear optics [20], amongst others. In this context, the EPE provides an effective description for the time-dependence of the relevant spatially dependent field, typically being associated with its width both in the BEC [21, 22] and in optical settings [23]. The mathematical analysis and structure of the EPE have been extensively discussed in [24-27].

Note that for generic values of the coefficients, Equation (1) is equivalent to a third order autonomous dynamical system, which generically admits no closed form general solution [2]. The mathematical properties and applications of particular forms of Equation (1) have been widely investigated, such as the partial integrability of the anharmonic oscillator [2], the time-dependent driven anharmonic oscillator and its adiabaticity properties [28], toroidal p -branes, anharmonic oscillators, and (hyper) elliptic solutions [29], conformal mappings and other power series methods for solving ordinary differential equations [30], and the anharmonic oscillator in the context of the optimized basis expansion

[31]. The Painlevé analysis of Equation (1) was performed in [32]. Specific transformation properties of the anharmonic oscillator were considered in [1], where an excellent review of the Lie symmetries approach to Equation (3) can also be found.

The most general conditions on the functions f_1 , f_2 , and f_3 , for which Equation (3) may be integrable, as well as conditions for the existence of Lie point symmetries, were obtained in [1]. Time-dependent first integrals were also constructed. The main results of [1] are that if $n \notin \{-3, -1, 0, 1\}$, then Equation (3) can be point transformed to an equation of the form $d^2X/dT^2 + X^n(T) = 0$, can be linearized as $d^2X/dT^2 + k_2 = 0$, $k_2 \in \mathfrak{R} \setminus 0$, and it admits a two-dimensional Lie point symmetry algebra.

It is the purpose of the present paper to obtain, by using the results of [1], some classes of exact solutions of the anharmonic oscillator equation (1) without the forcing term. The first solution is obtained by considering a particular solution of the point transformed equation $d^2X/dT^2 + X^n(T) = 0$, equivalent to the initial anharmonic oscillator equation. The integrability condition obtained in [1] for the anharmonic oscillator can be formulated in terms of a Riccati equation for the $f_1(t)$ and $\frac{1}{f_3(t)} \frac{df_3}{dt}$ terms, respectively. By imposing some specific constraints on the coefficients of the Riccati equation, namely, by requiring that the Riccati equation can be reduced to a Bernoulli equation, two distinct classes of exact solutions of the anharmonic oscillator equation with zero forcing term are obtained. In the analysis outlined below, we shall use the generalized Sundman transformations $X(T) = F(t, x)$ and $dT = G(t, x)dt$ [33-35]. The latter have been widely applied in the literature [33, 34], namely, in the study of the mathematical properties of the second order differential equations, and the third order differential equation

$$\frac{d^3x}{dt^3} + h_1(t)\frac{d^2x}{dt^2} + h_2(t)\frac{dx}{dt} + h_3(t)x + h_4(t) = 0. \quad (2)$$

The present paper is organized as follows. Three distinct classes of general solutions of Equation (1) without the forcing term, which explicitly depict the time evolution of the anharmonic oscillator, are presented in Section 2. We discuss and conclude our results in Section 3.

2. Exact Integrability Cases for the Anharmonic Oscillator

In the present section, by starting from the integrability condition of the anharmonic oscillator equation obtained in [1], we obtain three cases of exact integrability of the anharmonic oscillator without forcing.

2.1. The integrability condition for the anharmonic oscillator

In the following, we assume that the forcing term $f_4(t)$ vanishes in Equation (1). Hence the latter takes the form

$$\frac{d^2x}{dt^2} + f_1(t)\frac{dx}{dt} + f_2(t)x + f_3(t)x^n = 0. \quad (3)$$

An integrability condition of Equation (3) can be formulated as the following:

Theorem 1 ([1]). *If and only if $n \notin \{-3, -1, 0, 1\}$, and the coefficients of Equation (3) satisfy the differential condition*

$$\begin{aligned} f_2(t) = & \frac{1}{n+3} \frac{1}{f_3(t)} \frac{d^2f_3}{dt^2} - \frac{n+4}{(n+3)^2} \left[\frac{1}{f_3(t)} \frac{df_3}{dt} \right]^2 \\ & + \frac{n-1}{(n+3)^2} \left[\frac{1}{f_3(t)} \frac{df_3}{dt} \right] f_1(t) + \frac{2}{n+3} \frac{df_1}{dt} + \frac{2(n+1)}{(n+3)^2} f_1^2(t), \end{aligned} \quad (4)$$

with the help of the pair of transformations

$$X(T) = Cx(t)f_3^{\frac{1}{n+3}}(t)e^{\frac{2}{n+3}\int^t f_1(\phi)d\phi}, \quad (5)$$

$$T(x, t) = C^{\frac{1-n}{2}} \int^t \frac{2}{f_3^{n+3}}(\xi) e^{\left(\frac{1-n}{n+3}\right) \int^\xi f_1(\phi) d\phi} d\xi, \quad (6)$$

where C is an arbitrary constant, Equation (3) can be point transformed into the second order differential equation for $X(T)$,

$$\frac{d^2 X}{dT^2} + X^n(T) = 0. \quad (7)$$

The general solution of Equation (7) is given by

$$T = T_0 + \epsilon \int \frac{dX}{\sqrt{2\left(C_0 - \frac{X^{n+1}}{n+1}\right)}}, \quad n \neq -1, \quad (8)$$

where T_0 and C_0 are arbitrary constants of integration. For convenience, we have denoted $T = T_{\pm}$, $C_0 = C_{0\pm}$, $T_0 = T_{0\pm}$, and $\epsilon = \pm$.

By substituting the integrability condition given by Equation (4) into Equation (3), we obtain the following integrable differential equation:

$$\begin{aligned} \frac{d^2 x}{dt^2} + f_1(t) \frac{dx}{dt} + \left\{ \frac{1}{n+3} \frac{1}{f_3(t)} \frac{d^2 f_3}{dt^2} - \frac{n+4}{(n+3)^2} \left[\frac{1}{f_3(t)} \frac{df_3}{dt} \right]^2 \right. \\ \left. + \frac{n-1}{(n+3)^2} \left[\frac{1}{f_3(t)} \frac{df_3}{dt} \right] f_1(t) + \frac{2}{n+3} \frac{df_1}{dt} + \frac{2(n+1)}{(n+3)^2} f_1^2(t) \right\} x + f_3(t) x^n = 0, \\ n \notin \{-3, -1, 0, 1\}. \end{aligned} \quad (9)$$

2.2. A particular exact solution for the anharmonic oscillator equation

The general solution of Equation (7) can be given as

$$T = T_0 + \frac{\epsilon}{C_0} X \sqrt{\frac{C_0(n+1) - X^{n+1}}{2(n+1)}} {}_2F_1 \left[1, \frac{n+3}{2(n+1)}; \frac{n+2}{n+1}; \frac{X^{n+1}}{C_0(n+1)} \right], \quad n \neq -1, \quad (10)$$

where ${}_2F_1(a, b; c; d)$ is the hypergeometric function. A particular solution of Equation (7) is given by

$$X(T) = [\epsilon(T - T_0)]^{\frac{2}{1-n}} \left[-\frac{(n-1)^2}{2(n+1)} \right]^{\frac{1}{1-n}}, \quad (11)$$

where we have defined $X(T) = X_{\pm}(T)$, and we have taken the arbitrary constant of integration as zero, $C_0 = 0$. In order to have a real value of the displacement $x(t)$, one must impose the condition $n < -1$ on the anharmonicity exponent n . From Equations (5) and (11), we obtain the result

$$x(t) = \frac{1}{C} [\epsilon(T - T_0)]^{\frac{2}{1-n}} \left[-\frac{(n-1)^2}{2(n+1)} \right]^{\frac{1}{1-n}} f_3^{-\frac{1}{n+3}}(t) e^{-\frac{2}{n+3} \int^t f_1(\phi) d\phi}, \quad (12)$$

where we have denoted $x(t) = x_{\pm}(t)$, for simplicity.

By inserting Equation (6) into Equation (12) yields the general solution of Equation (9) describing the time evolution of anharmonic oscillator. Therefore, we have obtained the following:

Corollary 1. *The anharmonic oscillator equation (9) has the particular solution*

$$x(t) = x_0 \left[C^{\frac{1-n}{2}} \int^t f_3^{\frac{2}{n+3}}(\xi) e^{\frac{(1-n)}{n+3} \int^{\xi} f_1(\phi) d\phi} d\xi - T_0 \right]^{\frac{2}{1-n}} f_3^{-\frac{1}{n+3}}(t) e^{-\frac{2}{n+3} \int^t f_1(\phi) d\phi}, \quad (13)$$

$$n < -1,$$

where we have defined $x_0 = C^{-1} \left[-\frac{(n-1)^2}{2(n+1)} \right]^{\frac{1}{1-n}}$.

2.3. Second integrability case for the anharmonic oscillator equation

Now, by rearranging the terms of Equation (4) yields the following Riccati equation for $f_1(t)$ given by:

$$\frac{df_1}{dt} = a(t) + b(t)f_1(t) + c(t)f_1^2(t), \quad (14)$$

where the coefficients $a(t)$, $b(t)$, and $c(t)$ are defined as

$$a(t) = \frac{3+n}{2} f_2(t) - \frac{1}{2f_3(t)} \frac{d^2 f_3}{dt^2} + \frac{4+n}{2(3+n)} \left[\frac{1}{f_3(t)} \frac{df_3}{dt} \right]^2, \quad (15)$$

$$b(t) = \frac{1-n}{2(3+n)} \left[\frac{1}{f_3(t)} \frac{df_3}{dt} \right], \quad (16)$$

$$c(t) = -\frac{1+n}{3+n}. \quad (17)$$

We consider now that the coefficient $a(t)$ of Equation (14) vanishes, so that Equation (15) can be written as

$$f_2(t) = \frac{1}{3+n} \frac{1}{f_3(t)} \frac{d^2 f_3}{dt^2} - \frac{4+n}{(3+n)^2} \left[\frac{1}{f_3(t)} \frac{df_3}{dt} \right]^2. \quad (18)$$

Hence the Riccati equation (14) becomes a Bernoulli type equation, with the general solution given by

$$f_1(t) = \frac{f_3^{\frac{1-n}{2(3+n)}}(t)}{C_1 + \frac{1+n}{3+n} \int^t f_3^{\frac{1-n}{2(3+n)}}(\phi) d\phi}, \quad (19)$$

where C_1 is an arbitrary constant of integration.

By substituting Equations (18) and (19) into Equation (3), the latter yields the following differential equation:

$$\begin{aligned} \frac{d^2 x}{dt^2} + \left[\frac{f_3^{\frac{1-n}{2(3+n)}}(t)}{C_1 + \frac{1+n}{3+n} \int^t f_3^{\frac{1-n}{2(3+n)}}(\phi) d\phi} \right] \frac{dx}{dt} \\ + \left\{ \frac{1}{3+n} \frac{1}{f_3(t)} \frac{d^2 f_3}{dt^2} - \frac{4+n}{(3+n)^2} \left[\frac{1}{f_3(t)} \frac{df_3}{dt} \right]^2 \right\} x + f_3(t) x^n = 0. \quad (20) \end{aligned}$$

Therefore, we have obtained the following:

Corollary 2. *The general solution of Equation (20), describing the time evolution of the anharmonic oscillator, is given by*

$$x(t) = x_0 \left[C^{\frac{1-n}{2}} \int^t \frac{2}{f_3^{n+3}}(\xi) e^{(\frac{1-n}{n+3}) \int^\xi \left[\frac{f_3^{\frac{1-n}{2(3+n)}}(\phi)}{C_1 + \frac{1+n}{3+n} \int^\phi f_3^{\frac{1-n}{2(3+n)}}(\psi) d\psi} \right] d\phi} d\xi - T_0 \right]^{\frac{2}{1-n}}$$

$$\times f_3^{-\frac{1}{n+3}}(t) e^{-\frac{2}{n+3} \int^t \left[\frac{f_3^{\frac{1-n}{2(3+n)}}(\phi)}{C_1 + \frac{1+n}{3+n} \int^\phi f_3^{\frac{1-n}{2(3+n)}}(\xi) d\xi} \right] d\phi}, \quad n \notin \{-3, -1, 0, 1\}. \quad (21)$$

2.4. Third integrability case for the anharmonic oscillator equation

Now, by introducing a new function $u(t)$ defined as

$$u(t) = \frac{1}{f_3(t)} \frac{df_3}{dt}, \quad (22)$$

or equivalently,

$$f_3(t) = f_{03} e^{\int^t u(\phi) d\phi}, \quad (23)$$

where f_{03} is an arbitrary constant of integration, after substituting $u(t)$ into Equation (4) yields the following Riccati equation for $u(t)$, given by:

$$\frac{du}{dt} = a_1(t) + b_1(t)u(t) + c_1(t)u^2(t), \quad (24)$$

where the coefficients are defined as

$$a_1(t) = (3+n)f_2(t) - \frac{2(1+n)}{3+n} f_1^2(t) - 2 \frac{df_1}{dt}, \quad (25)$$

$$b_1(t) = \frac{1-n}{3+n} f_1(t), \quad (26)$$

$$c_1(t) = \frac{1}{3+n}. \quad (27)$$

We consider that the coefficient $a_1(t)$ of Equation (24) vanishes, as before, so that Equation (25) can be written as

$$f_2(t) = \frac{2(1+n)}{(3+n)^2} f_1^2(t) + \frac{2}{(3+n)} \frac{df_1}{dt}. \quad (28)$$

Then the Riccati equation (24) becomes a Bernoulli type equation, with the general solution given by

$$u(t) = \frac{e^{\frac{1-n}{3+n} \int^t f_1(\phi) d\phi}}{C_2 - \frac{1}{3+n} \int^t e^{\frac{1-n}{3+n} \int^\xi f_1(\phi) d\phi} d\xi}, \quad (29)$$

where C_2 is an arbitrary constant of integration. Thus, the coefficient $f_3(t)$ of Equation (3) is readily given by

$$f_3(t) = f_{03} e^{\int^t \left[\frac{e^{\frac{1-n}{3+n} \int^\psi f_1(\phi) d\phi}}{C_2 - \frac{1}{3+n} \int^\psi e^{\frac{1-n}{3+n} \int^\xi f_1(\phi) d\phi} d\xi} \right] d\psi}. \quad (30)$$

By substituting Equations (28) and (30) into Equation (3), the latter yields the following differential equation:

$$\begin{aligned} \frac{d^2x}{dt^2} + f_1(t) \frac{dx}{dt} + \left[\frac{2(1+n)}{(3+n)^2} f_1^2(t) + \frac{2}{(3+n)} \frac{df_1}{dt} \right] x \\ + f_{03} e^{\int^t \left[\frac{e^{\frac{1-n}{3+n} \int^\psi f_1(\phi) d\phi}}{C_2 - \frac{1}{3+n} \int^\psi e^{\frac{1-n}{3+n} \int^\xi f_1(\phi) d\phi} d\xi} \right] d\psi} x^n = 0. \end{aligned} \quad (31)$$

Therefore, we have obtained the following:

Corollary 3. *The general solution of Equation (31), describing the time evolution of an anharmonic oscillator, is given by*

$$\begin{aligned}
 x(t) = x_0 f_{03}^{-\frac{1}{3+n}} & \left[C^{\frac{1-n}{2}} f_{03}^{\frac{2}{3+n}} \int^t e^{\frac{2}{3+n} \int^\xi \left[\frac{\frac{1-n}{e^{3+n}} \int^\psi f_1(\phi) d\phi}{C_2 - \frac{1}{3+n} \int^\psi \frac{1-n}{e^{3+n}} \int^\rho f_1(\phi) d\phi} + \frac{1-n}{2} f_1(\psi) \right] d\psi} d\xi - T_0 \right]^{\frac{2}{1-n}} \\
 & \times e^{-\frac{1}{3+n} \int^t \left[\frac{\frac{1-n}{e^{3+n}} \int^\psi f_1(\phi) d\phi}{C_2 - \frac{1}{3+n} \int^\psi \frac{1-n}{e^{3+n}} \int^\rho f_1(\phi) d\phi} + 2f_1(\psi) \right] d\psi}, \quad n \notin \{-3, -1, 0, 1\}. \quad (32)
 \end{aligned}$$

3. Conclusion

In the limit of a small function $X(T)$, and by assuming that the constant n is large, $n \rightarrow +\infty$, in view of Equation (7), we obtain a linear relation between $X(T)$ and $T(t)$, given by

$$X(T) = \epsilon \sqrt{2C_0} (T - T_0). \quad (33)$$

With the help of Equations (5) and (6), the approximate solution of Equation (3), describing the time evolution of anharmonic oscillator is given by

$$x(t) \approx \frac{\epsilon \sqrt{2C_0}}{C} \left[C^{\frac{1-n}{2}} \int^t f_3^{\frac{2}{n+3}}(\xi) e^{\left(\frac{1-n}{n+3}\right) \int^\xi f_1(\phi) d\phi} d\xi - T_0 \right] f_3^{-\frac{1}{n+3}}(t) e^{-\frac{2}{n+3} \int^t f_1(\phi) d\phi}. \quad (34)$$

With this approximate solution, once the functions $f_1(t)$ and $f_3(t)$ are given, one can study the time evolution of the anharmonic oscillator for small $X(t)$ and for a very large anharmonicity exponent n .

In the present paper, by extending the results of [1], where the first integral of Equation (3) was obtained, we have obtained three classes of exact general solutions of Equation (3), by explicitly showing that the theorem obtained in [1] is very useful for obtaining the explicit general solutions of the anharmonic oscillator type second order differential equations.

In order to have real solutions, the general solutions equations (13), (21), and (32) of the second order differential equations (9), (20), and (31), respectively, must obey the condition $n < -1$, thus leading to an anharmonic term of the form $f_3(t)/x^n$, $n > 0$. Such a term may be singular at $x = 0$. Note that in [36], the author has used the degree theory to remove a technical assumption in the non-resonance results in [37] and to obtain a complete set of the non-resonance conditions for differential equations with repulsive singularities. In doing so, a nice relation between the Hill's equation $\frac{d^2x}{dt^2} + \eta(t)x(t) = 0$ and the EPE was established. This relation itself is useful in studying the stability of periodic solutions of Lagrangian systems of degree of freedom of $3/2$.

It is well-known that the second order ordinary differential equations with anharmonic term of the form $f_3(t)/x^n$ entail many problems in the applied sciences. Some examples are the Brillouin focusing system, and the motion of an atom near a charged wire. The Brillouin focusing system can be described by the second order differential equation

$$\frac{d^2x}{dt^2} + \alpha(1 + \cos t)x(t) = \frac{\beta}{x(t)}, \quad (35)$$

where α and β are positive constants. In the context of electronics, this differential equation governs the motion of a magnetically focused axially symmetric electron beam under the influence of a Brillouin flow, as shown in [7]. From the mathematical point of view, this differential equation is a singular perturbation of a Mathieu equation

$$\frac{d^2x}{dt^2} + (a - 2q \cos 2t)x(t) = 0, \quad (36)$$

where a and q are arbitrary constants. Existence and uniqueness of elliptic periodic solutions of the Brillouin electron beam focusing system has been discussed in [8]. Hence, the results obtained in the present paper could open the possibility of obtaining some exact solutions of non-linear differential equations of scientific or technological interest.

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